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INTERACTIVE METHOD FOR SOLVING MULTI-OBJECTIVE CONVEX PROGRAMS

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Abstract. In this paper an interactive procedure for solving multi-objective programming problem has been developed. Algorithm for the convex case is based on some elements of the theory of convex programming irrespective of any constraint qualification. Sensitivity analysis of the convex program by marginal value formula from the input optimization provides necessary information to help the user in the dialog part of the procedure.

Key words and phrases: multi-objective programming, convex programming, interactive algorithms, constraint qualifications, marginal value

1. INTRODUCTION

Let us consider multi-objective program

$$\text{MOP}\qquad \min\{f^j(z),\ j \in J\}\quad \text{s.t.}\quad z \in Z = \{z : g^k(z) \leq 0,\ k \in K\}$$

where $f^j : \mathbf{R}^n \to \mathbf{R}$, $j \in J = \{1, \dots, q\}$, $q \geq 2$ are objective functions and $g^k : \mathbf{R}^n \to \mathbf{R}$, $k \in K = \{1, \dots, r\}$ are constraint functions. Suppose that the feasible set $Z \subset \mathbf{R}^n$ is nonempty. The multi-objective function will be denoted by $f(z) = (f^1(z), \dots, f^q(z))$, i.e. $f : \mathbf{R}^n \to \mathbf{R}^q$. When all functions f^j, $j \in J$ and g^k, $k \in K$ are convex, such program is termed a multi-objective convex program.

The solutions of a multi-objective program are referred to as Pareto-optimal (noninferior, efficient or nondominated) solutions.

DEFINITION 1. A point $z \in Z$ is a Pareto-minimum of MOP if there is no other $z \in Z$ such that $f^j(z) \leq f^j(z^*)$, $j \in J$ with at least one strict inequality "<".

Since a Pareto-optima set usually contains many solutions that are nonequivalent and noncomparable, the question is how to choose the final solution from this set. Any solution that satisfactorily terminates the decision process is called a final solution. In the interactive approach the user actively participates in the solution process by supplying preference information through the dialog (see e.g. [1], [3], [7], [8], [11]). The paper is organized as follows: An interactive method for solving multi-objective program is proposed in Section 2. This rather general procedure is based on the idea to achieve improvement of some objectives by relaxing the others. Modifications of the algorithm for a convex case are described in Section 3. The choice of objectives for relaxation is made by using properties of the minimal index set of active constraints. The relaxation levels are assigned by using information provided by the marginal value formula from input optimization. The method is illustrated by a numerical example in Section 4.

2. DESCRIPTION OF THE INTERACTIVE METHOD

We develop an interactive method that explores the Pareto-optima set in search of the acceptable solution. The algorithm below does not require convexity neither continuity assumption.

INITIALIZATION. Choose an arbitrary feasible point $z^0 \in Z$. Set $y = z^0$, $h = 1$. The point y will be termed referential point.

STEP 1. Calculating Pareto-minimum.

Solve

$$P_y \qquad \begin{aligned} &\min \sum_{j \in J} f^j(z) \\ &\text{s.t.} \\ &f^j(z) \leq f^j(y), \qquad j \in J \\ &z \in Z. \end{aligned}$$

Then an optimal solution $\hat{z}^h$ of P_y is a Pareto-minimum of MOP (by Soland's characterization of Pareto-optimality [10]).

If $\hat{z}^h = y$, then the referential point y is a Pareto-minimum; otherwise, the solution $\hat{z}^h \neq y$ thus obtained is a Pareto-minimum "better" than the referential point y. Set $z^* = \hat{z}^h$.

STEP 2. Interaction with the user

Pareto-minimum z^* and the corresponding objective vector $f(z^*) = (f^1(z^*), \ldots, f^q(z^*))$ are offered to the user. If one is satisfied with all objective values $f^j(z^*)$, $j \in J$, then z^* is the solution for "realization". Stop.

If the user does not accept the solution offered, than one has to classify objectives into two groups I and R, where I is an index subset of objectives that the user wants to improve further, and R — an index subset of objectives that the user is willing to relax. Here $I \cup R = J$, $I \cap R = \emptyset$.

Moreover, the user has to assign amounts $\rho_j > 0$, $j \in R$ to relax the values $f^j(z^*)$, $j \in R$ in exchange for an improvement of some values $f^i(z^*)$, $i \in I$.

STEP 3. Calculating a new referential point

Solve

$$P_*(\rho) \qquad \begin{aligned} &\min_{(z)} \sum_{i \in I} f^i(z) \\ &\text{s.t.} \\ &f^i(z) \le f^i(z^*), \qquad i \in I \\ &f^j(z) \le f^j(z^*) + \rho_j, \qquad j \in R \\ &z \in Z. \end{aligned}$$

Here $\min_{(z)}$ means minimizing at the variable z for fixed parameter values ρ_j, $j \in R$. Let $\hat{y}^h$ be an optimal solution of the program $P_*(\rho)$, where $\rho = (\rho_j)$, $j \in R$ is the given vector. Set $y = \hat{y}^h$, $h = h + 1$ and go to Step 1. A point $\hat{y}^h$ is not necessarily Pareto-minimum of MOP.

NOTE. The weighted sum of objectives used in the algorithm can be replaced by a strictly increasing function of objectives.

The proposed algorithm is along a same line as STEM [1] and related methods based on the relaxation of objectives [3], [7], [8]. Unfortunately, the methods that attempt to improve some objectives usually do not give any guarantees of the improvement required. It is necessary to perform the complete optimization step in order to see if user's wishes are realistic (see [5]).

The important help to the user in expressing preference information would be to know in advance the behavior of objectives at the obtained Pareto-optimum. This kind of information is difficult to obtain in general nonlinear case, so we will consider this problem for convex multi-objective programs.

3. ALGORITHM FOR A CONVEX CASE

We study modification of the algorithm from the previous section applied to the search for satisfactory solution of the convex MOP.

Pareto-optimum in Step 1 and a new referential point from the Step 3 are obtained by solving convex programs P_y and $P_*(\rho)$ respectively. Slater's condition is not necessarily satisfied for such programs, so Kuhn-Tucker optimality conditions and the corresponding numerical methods are not applicable.

That is why we study how some elements of the "BBZ" theory of convex programming [2] irrespective of any constraint qualification can be used in the interactive procedure.

On the intersection of $\mathcal{F}_y = \{z : f^j(z) \le f^j(y),\ j \in J\}$ and the feasible set Z, define the index set of always active objectives and constraints:

$$J_y^= = \{j \in J : z \in \mathcal{F}_y \cap Z \Rightarrow f^j(z) = f^j(y)\}$$

and

$$K_y^= = \{k \in K : z \in \mathcal{F}_y \cap Z \Rightarrow g^k(z) = 0\}.$$

We introduce also the sets:

$$\mathcal{F}_y^= = \{z : f^j(z) = f^j(y),\ j \in J_y^=\}$$

and

$$Z_y^= = \{z : g^k(z) = 0, k \in K_y^=\}.$$

First we remind of constructive characterization of Pareto-optimality [5] that is a generalization of the result for unconstrained convex case from [2].

THEOREM 1. *A point $y \in Z$ is a Pareto-minimum of convex MOP if and only if $J_y^= = J$.*

The sets defined above are used in the algorithm as follows: At the point y the sets $J_y^=$ and $K_y^=$ are calculated. If $J_y^= = J$, then y is a Pareto-minimum. Set $z^* = y$.

If $J_y^= \neq J$, then Pareto-minimum z^*, "better" than y, has to be found. The feasible set $\mathcal{F}_y \cap Z$ of the program P_y is transformed as:

$$\mathcal{F}_y \cap Z = \{z : f^j(z) \leq f^j(y), j \in J \setminus J_y^=; g^k(z) \leq 0, k \in K \setminus K_y^=\} \cap \mathcal{F}_y^= \cap Z_y^=.$$

Such modified constraints satisfy Slater's condition relative to $\mathcal{F}_y^= \cap Z_y^=$. Program P_y can be solved now by any method, including those based on Kuhn-Tucker condition. Optimal solution z^* of P_y is a Pareto-optimum of MOP.

On the basis of Pareto-minimum z^* and corresponding $f^j(z^*)$, $j \in J$, the user has to classify objectives from J into two groups I and R. Very often further improvements of all objectives from I are not possible for any amount of relaxation of objectives from R, no matter how great it is. An analysis of the behaviour of objectives in the neighbourhood of the current z^* for the chosen pair (I, R) can be performed by:

$$I_*^= = \{i \in I : z \in \mathcal{F}_*^I \cap Z \Rightarrow f^i(z) = f^i(z^*)\}$$

where

$$\mathcal{F}_*^I = \{z : f^i(z) \leq f^i(z^*), i \in I\}.$$

The set $I_*^= \subset I$ contains indices of those objectives which remain unchangeable, i.e. equal to $f^i(z^*)$, $i \in I_*^=$ for any amount of relaxation of objectives from R.

For different set pairs (I, R) the index sets $I_*^=$ and $I_*^< = I \setminus I_*^=$ are calculated. Useful advice in choosing the pairs (I, R) can be to start with set R of minimal cardinality because the user attempts to improve as many objectives as possible. There are at most $2^q - 2$ possible choices because I can be any subset of J, but not an empty set or the set J itself.

If $I_*^= = I$, then no objective value $f^i(z^*)$, $i \in I$ can be improved. If $I_*^= = \emptyset$, then all $f^i(z^*)$, $i \in I$ can be improved simultaneously.

The user classifies objectives on the basis of objective values $f^j(z^*)$, $j \in J$ and information on the sets $I_*^=$. If the improvements cannot possibly be achieved for any pair (I, R), then there is no better solution, with the given preference structure, than the current Pareto-solution z^*. Stop. On the other hand, if some of the required improvements are achievable for the pair (I, R), then the user assigns amounts $\rho_j > 0$, $j \in R$ to increase objectives from R.

We introduce notation: $\mathcal{F}_*^R(\rho) = \{z : f^j(z) \leq f^j(z^*) + \rho_j, j \in R\}$, and $K_*^= = \{k \in K : z \in \mathcal{F}_*(\rho) \cap Z \Rightarrow g^k(z) = 0\}$. The feasible set of the program $P_*(\rho)$ is rewritten as:

$$\begin{aligned}\mathcal{F}_*(\rho) = \mathcal{F}_*^I \cap \mathcal{F}_*^R(\rho) \cap Z = \{z : & f^i(z) \leq f^i(z^*), i \in I \setminus I_*^=;\\ & f^j(z) \leq f^j(z^*) + \rho_j, j \in R;\\ & g^k(z) \leq 0, k \in K \setminus K_*^=\} \cap (\mathcal{F}_*^I)^= \cap Z_*^=,\end{aligned}$$

where $(\mathcal{F}_*^I)^= = \{z : f^i(z) = f^i(z^*), i \in I_*^=\}$.

Then Slater's condition, relative to the set $(\mathcal{F}_*^I)^= \cap Z_*^=$, is satisfied for such modified constraints of $P_*(\rho)$.

With the given (I, R), $\rho_j > 0$, $j \in R$ and modified constraint set, go to Step 3 of the algorithm and find new referential point y by solving convex program $P_*(\rho)$.

NOTE. Modification of the feasible set, as well as the given characterization of Pareto-optimality have practical sense only if the objects of convex analysis used here can be calculated. Algorithms from [6], [14] are developed for the class of programs with faithfully convex functions (see [9]).

Improvement amounts of particular objectives from I depend directly on the given $\rho_j > 0$, $j \in R$. The problem of setting the relaxation levels is not studied systematically in the literature except when one objective is relaxed. The new marginal value formula recently proposed [12] offers mathematical tool for sensitivity analysis even in case of relaxing many objectives. Information resulting from sensitivity analysis serves as a support to the user in assigning $\rho_j > 0$, $j \in R$.

The marginal value formula is stated for the convex model

$$P(\theta) \qquad \begin{aligned}&\min_{(x)} f(x, \theta)\\ &\text{s.t.}\\ &g^k(x, \theta) \leq 0, \qquad k \in K = \{1, \ldots, r\}\end{aligned}$$

where $x \in \mathbf{R}^n$ is a decision variable, $\theta \in \mathbf{R}^p$ is a parameter and $f, g^k : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}$, $k \in K$ are continuous functions and $f(\cdot, \theta)$, $g^k(\cdot, \theta) : \mathbf{R}^n \to \mathbf{R}$, $k \in K$ are convex for every θ.

At every θ, denote by $X(\theta) = \{x : g^k(x, \theta) \leq 0, k \in K\}$ the feasible set, $\hat{X}(\theta) = \{\hat{x}(\theta)\}$ is the set of optimal solutions and $\hat{f}(\theta) = f(\hat{x}(\theta), \theta)$ is the optimal value.

In many applications it is desirable the continuity of this triple around some $\theta = \theta^*$. Region of stability at θ^* is the set $S \subset \mathbf{R}^n$ where point-to-set mapping $X : \theta \to X(\theta)$ is lower semicontinuous at θ^* (in Hogan's sense [4]). Perturbations of θ from θ^* over the region S guarantee upper semicontinuity of the mapping $\hat{X} : \theta \to \hat{X}(\theta)$ and continuity of $\hat{f}$, provided that $\hat{X}(\theta^*) \neq \emptyset$ and bounded (see [15]).

The problem of calculating the marginal value

$$\lim_{\theta\in S,\,\theta\to\theta^*} \frac{\hat{f}(\theta)-\hat{f}(\theta^*)}{||\theta-\theta^*||}$$

on a region of stability has been studied in [**12**], [**15**], [**16**]. The formula from [**12**] is stated for convex model in terms of the special Lagrange function

$$L_*^{<}(x,u;\theta) = f(x,\theta) + \sum_{k\in K\setminus K^{=}(\theta^*)} u_k g^k(x,\theta)$$

on every region of stability where point-to-set mapping $X_*^{=} : \theta \to X_*^{=}(\theta)$, defined by $X_*^{=} = \{x : g^k(x,\theta) \le 0, k \in K^{=}(\theta^*)\}$, is lower semicontinuous.

THEOREM 2 ([**12**]). *Consider the convex model $P(\theta)$ with differentiable functions at some θ^*, with $\hat{X}(\theta^*) \neq \varnothing$ and bounded. Let S be an arbitrary region of stability at θ^* where the mapping $X_*^{=}$ is lower semicontinuous at θ^*. Suppose that the saddle point $(\hat{x}(\theta^*), \hat{u}_k(\theta^*) : k \in K \setminus K^{=}(\theta^*))$ is unique and that the gradients $\nabla f(x,\theta)$, $\nabla g^k(x,\theta)$, $k \in K\setminus K^{=}(\theta^*)$ are continuous at $(\hat{x}(\theta^*),\theta^*)$. Then for every path $\theta \in S$, $\theta \to \theta^*$ and $\hat{x}(\theta) \to \hat{x}(\theta^*)$, for which the limits*

$$l = \lim_{\theta\in S,\,\theta\to\theta^*} \frac{\theta-\theta^*}{||\theta-\theta^*||} \quad \text{and} \quad z = \lim_{\theta\in S,\,\theta\to\theta^*} \frac{\hat{x}(\theta)-\hat{x}(\theta^*)}{||\theta-\theta^*||}$$

exist, the marginal value is

$$\lim_{\theta\in S,\,\theta\to\theta^*} \frac{\hat{f}(\theta)-\hat{f}(\theta^*)}{||\theta-\theta^*||} = \nabla_\theta L_*^{<}(\hat{x}(\theta^*),\hat{u}(\theta^*);\theta^*)\cdot l + \nabla_x L_*^{<}(\hat{x}(\theta^*),\hat{u}(\theta^*);\theta^*)\cdot z.$$

The formula contains the term with a limit z — the directional derivative of the optimal solutions. The conditions for existence of this limit are considered in [**12**] and [**13**].

Let us consider now convex MOP at some Pareto-minimum z^*. Suppose that the classification (I,R) of objectives based on $f^j(z^*)$, $j \in J$ is given. Then $P_*(\rho)$ (from Step 3) can be considered as a convex model with a parameter vector $\rho = (\rho_j)$, $j \in R$.

Every ρ determines the feasible set

$$\mathcal{F}_*(\rho)\cap Z = \{z : f^i(z) \le f^i(z^*), i \in I, f^j(z) \le f^j(z^*) + \rho_j, j \in R\} \cap Z,$$

the set of optimal solutions $\{\hat{z}(\rho)\}$ and the optimal value $\hat{\varphi}(\rho) = \varphi(\hat{z}(\rho)) = \sum_{i\in I} f^i(\hat{z}(\rho))$ of the model $P_*(\rho)$.

Similarly as before, let

$$J_*^{=}(\rho) = \{j \in J : z \in \mathcal{F}_*(\rho)\cap Z \Rightarrow f^j(z) = f^j(z^*)\}$$

be the index set of active objectives on $\mathcal{F}_*(\rho)\cap Z$, and

$$K_*^{=}(\rho) = \{k \in K : z \in \mathcal{F}_*(\rho)\cap Z \Rightarrow g^k(z) = 0\}$$

the index set of active constraints on $\mathcal{F}_*(\rho) \cap Z$.

Consider model $P_*(\rho)$ for $\rho = \rho^0 = 0$. Then $J_*^=(\rho^0) = J$, by the characterization of Pareto-minimum z^* (Theorem 1). The value $\hat{\varphi}(\rho^0) = \sum_{i \in I} f^i(\hat{z}(\rho^0)) = \sum_{i \in I} f^i(z^*)$ is obviously improved for parameter perturbations over $S(\rho^0) = \{\rho : \rho_j > \rho_j^0 = 0, j \in R\}$ where $S(\rho^0)$ is the region of stability for model $P_*(\rho)$ at $\rho = \rho^0$.

Since $J \setminus J_*^=(\rho^0) = \emptyset$, the restrictive Lagrange function at ρ^0

$$\mathcal{L}_0^<(z,\mu;\rho) = \sum_{i \in I} f^i(z) + \sum_{k \in K \setminus k_*^=(\rho^0)} \mu_k g^k(z) = \mathcal{L}_0^<(z,\mu)$$

does not depend on ρ. Now Theorem 2 applied to the model $P_*(\rho)$ at ρ^0 yield:

COROLLARY 1. *Consider the convex model $P_*(\rho)$ with differentiable functions at ρ^0. Suppose that the saddle point $(\hat{z}(\rho^0), \hat{\mu}_k(\rho^0) : k \in K \setminus K_*^=(\rho^0))$ is unique and that the gradients $\nabla f^i(z)$, $i \in I$, $\nabla g^k(z)$, $k \in K \setminus K_*^=(\rho^0)$ are continuous at $\hat{z}(\rho^0)$. Then for every path $\rho \in S(\rho^0)$, $\rho \to \rho^0$ for which the limit*

$$\zeta^0 = \lim_{\rho \in S(\rho^0),\, \rho \to \rho^0} \frac{\hat{z}(\rho) - \hat{z}(\rho^0)}{\|\rho - \rho^0\|}$$

exists, it follows

$$\lim_{\rho \in S(\rho^0),\, \rho \to \rho^0} \frac{\hat{\varphi}(\rho) - \hat{\varphi}(\rho^0)}{\|\rho - \rho^0\|} = \nabla_z \mathcal{L}_0^<(\hat{z}(\rho^0), \hat{\mu}(\rho^0)) \cdot \zeta^0 \tag{1}$$

The formula (1) is used for constructing the path $\rho \in S(\rho^0)$ from ρ^0 along which satisfactory improvements of objectives from I will be achieved. Let $\pi(\rho^0) \subset S(\rho^0)$, $\rho \to \rho^0$ be an acceptable path thus found. The user is asked to choose on the path $\pi(\rho^0)$ a vector $\rho^1 = (\rho_j^1)$, $j \in R$, with small positive components that represent amounts of relaxation for objectives from R.

Further improvement of $\hat{\varphi}(\rho^1)$ is possible along some path from ρ^1 through the region of stability $S(\rho^1) = \{\rho : \rho_j > \rho_j^1, j \in R\}$.

Consider model $P_*(\rho)$ at $\rho = \rho^1$. Then

$$J_*^=(\rho^1) = I_*^=, \qquad \mathcal{F}_*^=(\rho^1) = \{z : f^i(z) = f^i(z^*), i \in J_*^=(\rho^1)\}$$
$$Z_*^=(\rho^1) = \{z : g^k(z) = 0, k \in K_*^=(\rho^1)\}.$$

Restrictive Lagrange function of $P_*(\rho)$ at ρ^1 is

$$\mathcal{L}_1^<(z,\nu;\rho) = \sum_{i \in I} f^i(z) + \sum_{j \in R} \nu_j [f^j(z) - f^j(z^*) - \rho_j]$$
$$+ \sum_{i \in I \setminus I_*^=} \nu_i [f^i(z) - f^i(z^*)] + \sum_{k \in K \setminus K_*^=(\rho^1)} \nu_k g^k(z).$$

Since $\Gamma_*^= = \{z : f^i(z) = f^i(z^*), i \in J_*^=(\rho^1); g^k(z) = 0, k \in K_*^=(\rho^1)\}$ does not depend on ρ, it is continuous at ρ^1 over $S(\rho^1)$. Marginal value formula at ρ^1 is obtained by application of Theorem 2.

COROLLARY 2. *Consider a convex model $P_\bullet(\rho)$ with differentiable functions at ρ^1. Suppose that the saddle point $(\hat{z}(\rho^1), \hat{\nu}_k(\rho^1) : k \in R \cup (I \setminus I_\bullet^=) \cup (K \setminus K_\bullet^=(\rho^1)))$ is unique and that gradients $\nabla f^j(z)$, $j \in J$, $\nabla g^k(z)$, $k \in K \setminus K_\bullet^=(\rho^1)$ are continuous at $\hat{z}(\rho^1)$. Then for every path $\rho \in S(\rho^1)$, $\rho \to \rho^1$ for which the limits*

$$l^1 = \lim_{\rho \in S(\rho^1),\, \rho \to \rho^1} \frac{\rho - \rho^1}{\|\rho - \rho^1\|} \quad \text{and} \quad \zeta^1 = \lim_{\rho \in S(\rho^1),\, \rho \to \rho^1} \frac{\hat{z}(\rho) - \hat{z}(\rho^1)}{\|\rho - \rho^1\|}$$

exist, the marginal value is

$$\lim_{\rho \in S(\rho^1),\, \rho \to \rho^1} \frac{\hat{\varphi}(\rho) - \hat{\varphi}(\rho^1)}{\|\rho - \rho^1\|} = \nabla_\rho \mathcal{L}_1^<(\hat{z}(\rho^1), \hat{\nu}(\rho^1); \rho^1) \cdot l^1 + \nabla_z \mathcal{L}_1^<(\hat{z}(\rho^1), \hat{\nu}(\rho^1); \rho^1) \cdot \zeta^1. \tag{2}$$

The other term, related to the derivative of the optimal solution, is equal to zero under the assumption that Slater's condition holds for the model $P_\bullet(\rho)$ at $\rho = \rho^1$ (see [16]) or for unconstrained multi-objective program, i.e. when $K = \varnothing$. In the later case

$$J_\bullet^=(\rho) = J_\bullet^=(\rho^1) \quad \text{and} \quad \mathcal{F}_\bullet^=(\rho) = \{z : f^i(z) = f^i(z^*), i \in J_\bullet^=(\rho^1)\}$$

for every $\rho \in S(\rho^1)$. Since

$$\nabla_\rho \mathcal{L}_1^<(\hat{z}(\rho^1), \hat{\nu}(\rho^1); \rho^1) = [-\hat{\nu}_j(\rho^1)], \qquad j \in R$$

marginal value formula (2) reduces to the previously known form:

$$\lim_{\rho \in S(\rho^1),\, \rho \to \rho^1} \frac{\hat{\varphi}(\rho) - \hat{\varphi}(\rho^1)}{\|\rho - \rho^1\|} = \nabla_\rho \mathcal{L}_1^<(\hat{z}(\rho^1), \hat{\nu}(\rho^1); \rho^1) \cdot l^1 = -\sum_{j \in R} \hat{\nu}_j(\rho^1) \cdot l_j^1. \tag{3}$$

4. EXAMPLE

The proposed method is illustrated by the program with five objectives:

$$\min_{z \in \mathbf{R}^4} \{f^1(z), \dots, f^5(z)\}$$

where:

$$f^1(z) = z_1^2 + z_2^2 - 2, \quad f^2(z) = (z_1 - 2)^2 + (z_2 - 2)^2 - 2, \quad f^3(z) = e^{z_3} + z_4^2 - 1,$$
$$f^4(z) = (z_3 - 1)^2 + z_4^2 - 1, \quad f^5(z) = -z_1 - z_2 + z_3 + 2.$$

Objective functions $f^j : \mathbf{R}^4 \to \mathbf{R}$, $j \in j = \{1, \dots, 5\}$ are faithfully convex.

INITIALIZATION. The initial point is $y^0 = (1, 1, 0, 2)^T$.

ITERATION $h = 1$

Step 1. For $y = y^0$ objective values vector is $f(y) = (0, 0, 4, 4, 0)^T$ and $J_y^= = \{1, 2, 5\}$. Since $J_y^= \neq J$, the point y is not a Pareto-minimum. Then the unique optimal solution $\hat{z}^1 = (1,1,0,0)^{\mathrm{T}}$ of the program P_y is a Pareto-minimum; set $z^* = \hat{z}^1$.

Step 2. The user has to classify objectives into two groups I and R on the basis of $f(z^*) = (0,0,0,0,0)^{\mathrm{T}}$. To help the user in this classification, sets $I_*^=$, $I \setminus I_*^=$ are calculated for different pairs (I, R) and results thus obtained constitute the Table 1. In order to study influence of the relaxation of one objective on the others, all sets R are taken to be singleton.

No. of the pair (I, R)	I	R	$I_*^=$	$I \setminus I_*^=$
1	$\{2,3,4,5\}$	$\{1\}$	$\{3,4\}$	$\{2,5\}$
2	$\{1,3,4,5\}$	$\{2\}$	$\{1,3,4,5\}$	$\varnothing$
3	$\{1,2,4,5\}$	$\{3\}$	$\{1,2,4,5\}$	$\varnothing$
4	$\{1,2,3,5\}$	$\{4\}$	$\{1.2\}$	$\{3,5\}$
5	$\{1,2,3,4\}$	$\{5\}$	$\{1,2,3,4\}$	$\varnothing$

Table 1.

It can be seen from the table that it is not possible to improve all remaining objectives by relaxing only one. Assume that the user is willing to relax the first objective and one is asked to assign the amount $\rho_1 > 0$ to be increased.

The solutions of $P_*(\rho)$ for different ρ are analysed. For every $\rho = \rho_1$, $\rho_1 > 0$ the optimal solution is $\hat{z}(\rho) = \left(\sqrt{1+\rho_1/2}, \sqrt{1+\rho_1/2}, 0, 0\right)^{\mathrm{T}}$. When $\rho^1 = \frac{21}{50}$, then $\hat{z}(\rho^1) = \left(\frac{11}{10}, \frac{11}{10}, 0, 0\right)^{\mathrm{T}}$ and corresponding $f(\hat{z}(\rho^1)) = (0.42, -0.38, 0, 0, -0.2)$. At $\rho^2 = \frac{5}{2}$, $\hat{z}(\rho^2) = \left(\frac{3}{2}, \frac{3}{2}, 0, 0\right)^{\mathrm{T}}$ and $f(\hat{z}(\rho^2)) = (2.5, -1.5, 0, 0, -1)$.

The $\rho_1^* = \frac{5}{2}$ is chosen as the satisfactory relaxation level. The new reference point $\hat{y}^1 = \hat{z}(\rho^2)$ is a solution of $P_*(\rho)$ with $\rho = \rho_1^*$. Set $y = \hat{y}^1$.

ITERATION $h = 2$

Step 1. Here $J_y^= = J$, so y is a Pareto-minimum. Set $z^* = y$.

Step 2. The objective values vector $f(z^*) = \left(\frac{5}{2}, -\frac{3}{2}, 0, 0, -1\right)^{\mathrm{T}}$ is offered to the user. Suppose that one tries to improve $f^4(z^*) = 0$. As Table 1 shows, worsening of one objective does not guarantee the improvement required. We are going to see if this is possible to achieve by worsening two objectives. That is why index sets $I_*^=$ and $I \setminus I_*^=$ are calculated for different sets R which consist of two elements.

According to Table 2, the best results are obtained with $R = \{1,3\}$. The sensitivity analysis will be carried out to help the user in assigning relaxation levels $\rho_1 > 0$ and $\rho_3 > 0$. Consider model $P_*(\rho)$ for $\rho = \rho^0 = (\rho_1^0 = 0, \rho_3^0 = 0)$. Then $\hat{z}(\rho^0) = z^* = \left(\frac{3}{2}, \frac{3}{2}, 0, 0\right)^{\mathrm{T}}$ and $\nabla_z \sum_{i \in I} f^i(\hat{z}(\rho^0)) = (-2, -2, -1, 0)$. For every $\rho \in S(\rho^0) = \{\rho = (\rho_1, \rho_3) : \rho_1 > 0, \rho_3 > 0\}$ optimal solution of $P_*(\rho)$ is $\hat{z}(\rho) = \left(\sqrt{9/4+\rho_1/2}, \sqrt{9/4+\rho_1/2}, \ln(1+\rho_3), 0\right)^{\mathrm{T}}$. Marginal value describes changes of $\hat{\varphi}(\rho)$ along different paths from ρ^0 through the region $S(\rho^0)$.

No. of the pair (I,R)	I	R	$I_*^=$	$I \setminus I_*^=$
1	$\{3,4,5\}$	$\{1,2\}$	$\{3,4\}$	$\{5\}$
2	$\{2,3,4\}$	$\{1,5\}$	$\{3,4\}$	$\{2\}$
3	$\{1,3,4\}$	$\{2,5\}$	$\{3,4\}$	$\{1\}$
4	$\{1,4,5\}$	$\{2,3\}$	$\{1,4,5\}$	$\varnothing$
5	$\{1,2,5\}$	$\{3,4\}$	$\{1,2\}$	$\{5\}$
6	$\{1,2,4\}$	$\{3,5\}$	$\{1,2\}$	$\{4\}$
7	$\{1,2,3\}$	$\{4,5\}$	$\{1,2\}$	$\{3\}$
8	$\{2,4,5\}$	$\{1,3\}$	$\varnothing$	$\{2,4,5\}$
9	$\{2,3,5\}$	$\{1,4\}$	$\varnothing$	$\{2,3,5\}$
10	$\{1,3,5\}$	$\{2,4\}$	$\varnothing$	$\{1,3,5\}$

Table 2.

On the path $\pi_1(\rho^0) = \{\rho \in S(\rho^0) : \rho = \rho^0 + \alpha(d_1,d_3)^{\mathrm{T}}, d_1, d_3 > 0, \alpha \geq 0\}$, $\zeta^1 = \dfrac{(d_1,d_1,6d_3,0)^{\mathrm{T}}}{6\sqrt{d_1^2+d_3^2}}$, so the marginal value calculated by formula (1) equals $MV_1 = \dfrac{-2d_1-3d_3}{3\sqrt{d_1^2+d_3^2}}$. On the path $\pi_2(\rho^0) = \{\rho \in S(\rho^0) : \rho = \rho^0 + (t,t^2)^{\mathrm{T}}, t > 0\}$ is $\zeta^2 = \left(\frac{1}{6},\frac{1}{6},0,0\right)^{\mathrm{T}}$ and $MV_2 = -\frac{2}{3}$. If the path is $\pi_3(\rho^0) = \{\rho \in S(\rho^0) : \rho = \rho^0 + (t^3,t^2)^{\mathrm{T}}, t > 0\}$, then $\zeta^3 = (0,0,1,0)^{\mathrm{T}}$ and $MV_3 = -1$.

The path $\pi_1(\rho^0)$ is chosen as acceptable and the point ρ^1 on this path. For $\rho^1 = \left(\rho_1^1 = \frac{13}{8}, \rho_3^1 = e^{1/4}-1\right)$ from the path $\pi_1(\rho^0)$, with $d_1 = \frac{13}{8}$, $d_3 = e^{1/4}-1$ and $\alpha = 1$, optimal solution is $\hat{z}(\rho^1) = \left(\frac{7}{4},\frac{7}{4},\frac{1}{4},0\right)^{\mathrm{T}}$ and the corresponding Lagrange multipliers are $\hat{\nu}_1(\rho^1) = \frac{3}{7}$, $\hat{\nu}_3(\rho^1) = 0.5e^{-1/4}$.

Different paths emanating from ρ^1 can be constructed through the region $S(\rho^1) = \{\rho = (\rho_1,\rho_3) : \rho_1 > \rho_1^1, \rho_3 > \rho_3^1\}$. We will analyse marginal value on three chosen paths. Along the path $\pi_1(\rho^1) = \{\rho \in S(\rho^1) : \rho = \rho^1 + \alpha(d_1,d_3)^{\mathrm{T}}, d_1, d_3 > 0, \alpha \geq 0\}$ we obtain $l^1 = \dfrac{(d_1,d_3)^{\mathrm{T}}}{\sqrt{d_1^2+d_3^2}}$. Marginal value for $\rho \in \pi_1(\rho^1)$, $\rho \to \rho^1$, calculated by formula (3), is equal to

$$MV_1 = \left(-\frac{3}{7}d_1 - 0.5e^{-1/4}d_3\right)\frac{1}{\sqrt{d_1^2+d_3^2}}.$$

On the path $\pi_2(\rho^1) = \{\rho \in S(\rho^1) : \rho = \rho^1 + (t,t^2)^{\mathrm{T}}, t > 0\}$ is $l^2 = (1,0)^{\mathrm{T}}$, then it follows $MV_2 = -3/7$. If the path has the form $\pi_3(\rho^1) = \{\rho \in S(\rho^1) : \rho = \rho^1 + (t^3,t^2)^{\mathrm{T}}, t > 0\}$, then $l^3 = (0,1)^{\mathrm{T}}$ and $MV_3 = -0.5e^{-1/4}$.

The point $\rho^2 = \left(\rho_1^2 = \frac{7}{2}, \rho_3^2 = e^{1/2}-1\right)$ from the path $\pi_1(\rho^1)$ is chosen as satisfactory. The modified program $P_*(\rho)$, with $\rho = \rho^2$ is solved and results are followed on Table 3.

The user chooses z^{it}, $it = 2$ as a new referential point. Set $y = z^{it}$.

it	z_1	z_2	z_3	z_4	f^1	f^2	f^3	f^4	f^5
0	1.5	1.5	0.	0.	2.5	−1.5	0.	0.	−1.
1	1.99999	2.	0.33333	0.	5.99999	−1.99999	0.39741	−0.55555	−1.66666
2	2.11666	1.87499	0.45833	0.	5.99589	−1.97076	0.58143	−0.70659	−1.53333

Table 3.

ITERATION $h = 3$

Step 1. For $y = (2.11666, 1.87499, 0.45833, 0.)^T$ objective values vector is $f(y) = (5.99589, -1.97076, 0.58143, -0.70659, -1.53333)^T$ and $J_y^= = \{3, 4\}$. Since $J_y^= \neq J$, y is not a Pareto-minimum. A few iterations of solving P_y by modified feasible direction method are presented:

it	z_1	z_2	z_3	z_4	$\sum_{j\in J} f^j(z)$
0	2.11666	1.87499	0.45833	0.	2.36664
1	1.95744	2.03617	0.45833	0.	2.32024
2	2.03129	1.96037	0.45833	0.	2.31326
3	1.97435	2.01757	0.45833	0.	2.31089
4	2.00136	1.99087	0.45833	0.	2.31003

Table 4.

Optimal solution of P_y is $\hat{z}^3 = (1.99583, 1.99583, 0.45833, 0.)^T$. This solution is a Pareto-minimum for the given problem, so $z^* = \hat{z}^3$.

Step 2. The user is satisfied with all reached objective values

$$f(z^*) = (5.96668, -1.99997, 0.58143, -0.70659, -1.53333)^T,$$

and $z^* = \hat{z}^3$ is the acceptable or the solution "for realization".

In this example the author took the role of the user and he gave preference information during the solution process.

5. CONCLUSION

The procedure proposed in this paper is along the same line as STEM method. The basic idea of STEM method — to improve some objectives by relaxing the others — is used and further developed in many methods which appeared later on. In comparison with some other procedures from this group, the user can try to improve more than one objective function and the achievement of the required improvements in the convex case are checked without necessarily performing complete optimization step.

The sensitivity analysis carried out by recently proposed marginal value formula from input optimization is an important support to the user in assigning the relaxation levels. It should be noted that the lack of information from the sensitivity analysis is referred to in the literature as a disadvantage of the known interactive methods.

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